The $n$-projective cotorsion pair of chain complexes

Marco Pérez
Department of Mathematics
Massachusetts Institute of Technology
maperez@mit.edu

October 13, 2014

Abstract
Let $\mathcal{P}_n(\text{Ch}(R))$ denote the class of $n$-projective chain complexes, i.e. those complexes whose projective dimension is at most $n$. We show that the cotorsion pair $(\mathcal{P}_n(\text{Ch}(R)), (\mathcal{P}_n(\text{Ch}(R)))^\perp)$ is cogenerated by a set (and so complete). Our proof consists in proving that every $n$-projective complex is filtered by the set of $n$-projective complexes with cardinality $\leq \kappa$, where $\kappa$ is a infinite cardinal greater that $\text{Card}(R)$. In the process, we show how to extend Enochs’ zig-zag argument from the category of modules to the category of chain complexes.

1 Preliminaries
1.1 Chain complexes

Let $R$ be any associative ring with identity. We use the following notation throughout these notes:

- $\mathcal{R}\text{Mod}$ denotes the category of left $R$-modules and homomorphisms.
- $\text{Ch}(R)$ denotes the category of chain complexes of left $R$-modules and chain maps.

Given a chain complex $X = (X_m)_{m \in \mathbb{Z}}$ with boundary maps $\partial^X_m : X_m \rightarrow X_{m-1}$, the module

$$Z_m(X) := \text{Ker}(\partial^X_m)$$

is called the $m$-cycle of $X$, and

$$B_m(X) := \text{Im}(\partial^X_{m+1})$$

the $m$-boundary of $X$. A chain complex $X$ is said to be exact if $Z_m(X) = B_m(X)$, for every $m \in \mathbb{Z}$. 

A chain map $f : X \to Y$ is a monomorphism (resp. epimorphism) if each $f_m : X_m \to Y_m$ is an injective (resp. surjective) homomorphism, or equivalently if the sequence

$$0 \to X \xrightarrow{f} Y \text{ (resp. } X \xrightarrow{f} Y \to 0)$$

is exact.

Given a chain complex $X$, a complex $X'$ is said to be a subcomplex of $X$ if there exists a monomorphism $i : X' \to X$. If $X'$ is a subcomplex of $X$, we define the quotient complex $X/X'$ as the complex whose components are given by

$$(X/X')_m = X_m/X'_m$$

and whose boundary maps $\partial_{m}^{X/X'} : X_m/X'_m \to X_{m-1}/X'_{m-1}$ are given by

$$\partial_{m}^{X/X'} : x + X'_m \mapsto \partial_{m}^X(x) + X'_{m-1}.$$

Given a chain map $f : X \to Y$, the image complex $\text{Im}(f)$ is the chain complex given by

$$(\text{Im}(f))_m := \text{Im}(f_m) = f_m(X_m),$$

whose boundary maps $\partial_{m}^{\text{Im}(f)} : f_m(X_m) \to f_{m-1}(X_{m-1})$ are defined by

$$\partial_{m}^{\text{Im}(f)} : x \mapsto \partial_{m}^X \circ f_{m}(x) = f_{m-1} \circ \partial_{m}^X(x).$$

The kernel complex $\text{Ker}(f)$ is the chain complex given by

$$(\text{Ker}(f))_m := \text{Ker}(f_m),$$

whose boundary maps $\partial_{m}^{\text{Ker}(f)} : \text{Ker}(f_m) \to \text{Ker}(f_{m-1})$ are defined by

$$\partial_{m}^{\text{Ker}(f)} : x \mapsto \partial_{m}^X \circ i_{m}(x),$$

where $i_{m}$ denotes the inclusion $\text{Ker}(f_m) \to X_m$.

### 1.2 Cotorsion pairs

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of chain complexes. The pair $(\mathcal{A}, \mathcal{B})$ is called a cotorsion pair if the following conditions are satisfied:

1. $\mathcal{A} = ^{\perp} \mathcal{B} := \{ X \in \text{Ch}(R) / \text{Ext}^1(X, B) = 0 \text{ for every } B \in \mathcal{B} \}.$
2. $\mathcal{B} = \mathcal{A}^{\perp} := \{ X \in \text{Ch}(R) / \text{Ext}^1(A, X) = 0 \text{ for every } A \in \mathcal{A} \}.$

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be complete if any of the two equivalent conditions holds:
(a) $(\mathcal{A}, \mathcal{B})$ has **enough projectives**: for every object $X$ there exist objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and a short exact sequence

$$0 \rightarrow B \rightarrow A \rightarrow X \rightarrow 0.$$  

(b) $(\mathcal{A}, \mathcal{B})$ has **enough injectives**: for every object $X$ there exist objects $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$, and a short exact sequence

$$0 \rightarrow X \rightarrow B' \rightarrow A' \rightarrow 0.$$  

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be **cogenerated** by a set $S \subseteq \mathcal{A}$ if $\mathcal{B} = S^\perp$.

The previous definitions also hold for left $R$-modules.

There is a wide range of complete cotorsion pairs, thanks to the following result, known as the Eklof and Trlifaj Theorem:

**Theorem 1.2.1.** [2, Theorem 10] Every cotorsion pair in $\mathcal{R}$-$\text{Mod}$ cogenerated by a set is complete.

The previous theorem is also valid for chain complexes.

### 2. The cotorsion pair $(\mathcal{P}_n(\text{Ch}(R)), (\mathcal{P}_n(\text{Ch}(R)))^\perp)$

#### 2.1. $n$-projective modules

Every left $R$-module $M$ has a projective resolution, i.e. an exact sequence

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where each $P_i$ is a projective module. The **projective dimension** of $M$ is defined as the integer

$$\text{pd}(M) := \min\{n \geq 0 : \text{Ext}^j(M, -) = 0, \forall j > n\},$$

provided such an integer exists. Otherwise, set $\text{pd}(M) = \infty$. It is known that $\text{pd}(M) \leq n$ if and only if there exists a finite projective resolution of length $n$, i.e. an exact sequence

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where $P_i$ is projective for every $0 \leq i \leq n$. A module $M$ is called **$n$-projective** if $\text{pd}(M) \leq n$. We denote the class of $n$-projective modules by $\mathcal{P}_n(\mathcal{R}\text{-Mod})$. The above comments and definition also hold for chain complexes, and we denote by $\mathcal{P}_n(\text{Ch}(R))$ the class of $n$-projective complexes.
Given a left $R$-module $M$, the $n$th disk complex centred at $M$ is the chain complex of the form

$$D^m(M) := \cdots \rightarrow 0 \rightarrow M \xrightarrow{=} M \rightarrow 0 \rightarrow \cdots$$

where $M$ appears at the $m$-th and $m-1$-th entries.

Suppose we are given an $n$-projective module $M$ with a projective resolution of length $n$ as above. By Eilenberg’s Trick, for each $i$ there exists a free module $F_i$ such that $P_i \oplus F_i \cong F_i$. Consider the disks complexes $D^{i+1}(F_i)$ where $0 \leq i \leq n-1$, and $D^n(F_n)$. Taking the direct sum of these disks and the given projective resolution of $M$, we get an exact sequence

$$0 \rightarrow P_n \oplus F_n \oplus F_{n-1} \rightarrow P_{n-1} \oplus F_n \oplus F_{n-1} \oplus F_n \rightarrow \cdots \rightarrow P_0 \oplus F_0 \rightarrow M \rightarrow 0$$

which turns out to be a free resolution of $M$.

From now on, let $\kappa$ be a fixed infinite cardinal such that $\kappa \geq \text{Card}(R)$. We shall say that a set $S$ is small if $\text{Card}(S) \leq \kappa$.

The following result is due to Aldrich et al. (see [1, Proposition 4.1]). It is a tool used to prove that $(\mathcal{P}_n, (\mathcal{P}_n)^\perp)$ is a complete cotorsion pair.

**Example 2.1.1.** The class $\mathcal{P}_n(R\text{Mod})$ is the left half of a complete and hereditary cotorsion pair $(\mathcal{P}_n(R\text{Mod}), (\mathcal{P}_n(R\text{Mod}))^\perp)$. Moreover, it is cogenerated by the set

$$S = \{ L \in \mathcal{P}_n(R\text{Mod}) : L \text{ is small} \}.$$

The reader can check the details in [3, Theorem 7.4.6] or in [1, Theorem 4.2].

### 2.2 $n$-projective complexes

The following definition are due to J. Gillespie [4, Definitions 3.3]: Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of left $R$-modules. Define $\widetilde{\mathcal{A}}$ as the class of all exact complexes $X$ such that $Z_m(X) \in \mathcal{A}$ (resp. $Z_m(X) \in \mathcal{B}$) for every $m \in \mathbb{Z}$.

**Proposition 2.2.1** (Gillespie, [4, Proposition 3.6]). If $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in $R\text{Mod}$, then $(\widetilde{\mathcal{A}}, (\widetilde{\mathcal{A}})^\perp)$ is a cotorsion pair in $\text{Ch}(R)$.

We know that $(\mathcal{P}_n(R\text{Mod}), (\mathcal{P}_n(R\text{Mod}))^\perp)$ is a complete cotorsion pair. So $(\mathcal{P}_n(R\text{Mod}), (\mathcal{P}_n(R\text{Mod}))^\perp)$ is a cotorsion pair in $\text{Ch}(R)$. To show $\mathcal{P}_n(\text{Ch}(R))$ is the left half of a complete cotorsion pair $(\widetilde{\mathcal{P}}_n(\text{Ch}(R)), (\mathcal{P}_n(\text{Ch}(R)))^\perp)$, we show that the classes $\mathcal{P}_n(R\text{Mod})$ and $\mathcal{P}_n(\text{Ch}(R))$ coincide and that the former pair is complete.
We first study the case \( n = 0 \). In this case we get the projective cotorsion pair \((\mathcal{P}_0(\text{RMod}), \widetilde{\text{RMod}})\), which induces a cotorsion pair \((\mathcal{P}_0(\text{RMod})), (\mathcal{P}_0(\text{RMod}))^\perp)\). The class \( \mathcal{P}_0(\text{RMod}) \) coincides with the class of projective chain complexes (See [5, Section 10.5]). It follows that \((\mathcal{P}_0(\text{Ch(R)})), (\mathcal{P}_0(\text{Ch(R)}))^\perp)\) is a complete cotorsion pair.

Now we study the case when \( n > 0 \). Before proving the completeness of \((\mathcal{P}_n, \text{dg}\mathcal{P}_n^\perp)\), we need some lemmas. The following lemma follows using induction and the fact that the class of exact chain complexes is closed under cokernel of monomorphisms in \(E\), i.e that if \(X \to Y\) is a monomorphism in \(\text{Ch}(R)\) with \(X\) and \(Y\) exact, then \(\text{CoKer}(X \to Y)\) is also exact.

**Lemma 2.2.1.** Let

\[
0 \to A^n \xrightarrow{f^n} A^{n-1} \to \cdots \to A^1 \xrightarrow{f^1} A^0 \xrightarrow{f^0} X \to 0
\]

be an exact sequence of chain complexes such that \(A^i\) is exact for every \(0 \leq i \leq n\). Then \(X\) is also exact.

**Lemma 2.2.2.** Consider a short exact sequence

\[
0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0
\]

of chain complexes over \(R\). Then the sequence

\[
0 \to Z_m(Y) \to Z_m(Z) \to Z_m(X) \to 0
\]

is exact if \(Y\) is an exact complex.

**Proof.** Let \(Z_m(f) : Z_m(Y) \to Z_m(Z)\) be the homomorphism induced by the universal property of kernels, given by \(y \mapsto f_m(y)\) for every \(y \in Z_m(Y)\). The homomorphism \(Z_m(g) : Z_m(Y) \to Z_m(X)\) is defined similarly. It is easy to check that \(Z_m(f)\) is monic and that \(\text{Ker}(Z_m(g)) = \text{Im}(Z_m(f))\). These facts do not depend on the exactness of \(Y\). Let \(x \in Z_m(X)\). There exists \(z \in Z_m\) such that \(x = g_m(z)\). We have \(g_{m-1} \circ \partial^Z_m(z) = \partial^X_m \circ g_m(z) = 0\). Since the sequence \(0 \to Y_m \to Z_m \to X_m \to 0\) is exact, there exists \(y \in Y_m\) such that \(\partial^Z_m(z) = f_{m-1}(y)\). Then \(f_{m-2} \circ \partial^Y_{m-1}(y) = \partial^Z_{m-1} \circ f_{m-1}(y) = 0\) and so \(\partial^Y_{m-1}(y) = 0\) since \(f_{m-2}\) is monic. By the exactness of \(Y\), there exists \(y' \in Y_m\) such that \(y = \partial^Y_m(y')\). Hence \(\partial^Z_m(z - f_m(y')) = 0\) and \(g_m(z - f_m(y')) = x\).

Using the previous lemma along with the induction principle, we obtain the following result.
Lemma 2.2.3. Let 
\[ 0 \to A^n \xrightarrow{f^n} A^{n-1} \to \cdots \to A^1 \xrightarrow{f^1} A^0 \to 0 \]
be an exact sequence of exact chain complexes. Then, for every \( m \in \mathbb{Z} \), there exists an exact sequence of modules
\[ 0 \to Z_m(A^n) \to Z_m(A^{n-1}) \to \cdots \to Z_m(A^1) \to Z_m(A^0) \to 0. \]

Proposition 2.2.2. A chain complex \( X \) is \( n \)-projective if, and only if, \( X \) is exact and each \( m \)-cycle is an \( n \)-projective module.

Proof. Let \( X \) be an exact complex with \( n \)-projective cycles. Consider a partial projective resolution
\[ 0 \to K \to P^{n-1} \to \cdots \to P^0 \to X \to 0. \]
Note \( K \) is exact by Lemma 2.2.1. Notice also that \( Z_m(P^i) \) is projective for every \( 0 \leq i \leq n-1 \) and every \( m \in \mathbb{Z} \). It follows that \( Z_m(K) \) is projective since \( Z_m(X) \) is \( n \)-projective. The converse follows similarly.

Therefore, \( \mathcal{P}_n(R \text{-mod}) \) is the class \( \mathcal{P}_n(\text{Ch}(R)) \) of \( n \)-projective chain complexes.

2.3 Small \( n \)-projective filtrations

Given a chain complex \( F \in \text{Ch}(R) \), we shall say that \( F \) is a free complex if \( F \) is exact and each \( Z_n(F) \) is a free left \( R \)-module. Note that every free complex is projective.

Lemma 2.3.1. Every free complex can be decomposed into a direct sum of free disks. Conversely, any direct sum of free disks is free.

Proof. Let \( F \) be a free complex. It is not hard to see that \( F \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(F)) \), where each \( D^{m+1}(Z_m(F)) \) is a free disk.

Now let \( F = \bigoplus_{m \in \mathbb{Z}} Z^{m+1}(F) \), where each \( F_m \) is a free module. It is clear that \( F \) is exact. Note that \( F \) has the form
\[ \cdots \to F_m \oplus F_{m+1} \xrightarrow{\partial^F_{m+1}} F_{m-1} \oplus F_m \xrightarrow{\partial^F_m} F_{m-2} \oplus F_{m-1} \to \cdots \]
where each boundary map \( \partial^F_m : F_{m-1} \oplus F_m \to F_{m-2} \oplus F_{m-1} \) is given by \((x, y) \mapsto (0, x)\). We have that \( Z_m(F) \cong F_m \) is a free module.
Proposition 2.3.1 (Eilenberg Trick’s in Ch(R)). If \( P \) is a projective complex, then there exists a free complex \( F \) such that \( P \oplus F \cong F \).

**Proof.** Let \( P \) be a projective module. Then \( P \) is isomorphic to a direct sum of projective disks \( P \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(P)) \). By Eilenberg Trick’s in \( R\)-Mod there exists a free module \( F_m \) such that \( Z_m(P) \oplus F_m \cong F_m \). It follows

\[
D^{m+1}(Z_m(P)) \oplus D^{m+1}(F_m) \cong D^{m+1}(F_m).
\]

Setting \( F = \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m) \), which is a free complex by the previous lemma, we obtain

\[
P \oplus F \cong \left( \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(P)) \right) \oplus \left( \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m) \right)
\cong \bigoplus_{m \in \mathbb{Z}} (D^{m+1}(Z_m(P)) \oplus D^{m+1}(F_m)) \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m) = F.
\]

\[\square\]

**Lemma 2.3.2.** Every \( n \)-projective complex has a free resolution of length \( n \).

**Proof.** We only prove the case \( n = 1 \). The general case can be proven similarly. Let \( X \in \mathcal{P}_1(\text{Ch}(R)) \) and let \( 0 \to P^1 \to P^0 \to X \to 0 \) be a projective resolution of \( X \) of length 1. By Eilenberg’s Trick in \( \text{Ch}(R) \), there exist free complexes \( F^0 \) and \( F^1 \) such that \( P^0 \oplus F^0 \cong F^0 \) and \( P^1 \oplus F^1 \cong F^1 \). Consider the short exact sequences

\[
0 \to F^1 \to F^1 \to 0 \to 0 \quad \text{and} \quad 0 \to F^0 \to F^0 \to 0 \to 0.
\]

Adding the sequences above, we get

\[
0 \to P^1 \oplus F^1 \oplus F^0 \to P^0 \oplus F^0 \oplus F^1 \to X \to 0 \cong 0 \to F^1 \oplus F^0 \to F^0 \oplus F^1 \to X \to 0.
\]

\[\square\]
Lemma 2.3.3. Given the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccccc}
0 & & 0 & & 0 & & 0 & \\
\cdots & \rightarrow & F_{m}^{n} \oplus F_{m+1}^{n} & \overset{\partial_{m+1}^{n}}{\rightarrow} & F_{m-1}^{n} \oplus F_{m}^{n} & \overset{\partial_{m}^{n}}{\rightarrow} & F_{m-2}^{n} \oplus F_{m-1}^{n} & \rightarrow \cdots \\
\cdots & \rightarrow & F_{m}^{n-1} \oplus F_{m+1}^{n-1} & \overset{\partial_{m+1}^{n-1}}{\rightarrow} & F_{m-1}^{n-1} \oplus F_{m}^{n-1} & \overset{\partial_{m}^{n-1}}{\rightarrow} & F_{m-2}^{n-1} \oplus F_{m-1}^{n-1} & \rightarrow \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & \rightarrow & F_{m}^{1} \oplus F_{m+1}^{1} & \overset{\partial_{m+1}^{1}}{\rightarrow} & F_{m-1}^{1} \oplus F_{m}^{1} & \overset{\partial_{m}^{1}}{\rightarrow} & F_{m-2}^{1} \oplus F_{m-1}^{1} & \rightarrow \cdots \\
\cdots & \rightarrow & F_{m}^{0} \oplus F_{m+1}^{0} & \overset{\partial_{m+1}^{0}}{\rightarrow} & F_{m-1}^{0} \oplus F_{m}^{0} & \overset{\partial_{m}^{0}}{\rightarrow} & F_{m-2}^{0} \oplus F_{m-1}^{0} & \rightarrow \cdots \\
\cdots & \rightarrow & X_{m+1} & \overset{\partial_{m+1}}{\rightarrow} & X_{m} & \overset{\partial_{m}}{\rightarrow} & X_{m-1} & \rightarrow \cdots \\
0 & & 0 & & 0 & & 0 & \\
\end{array}
\]

For every \( m \in \mathbb{Z} \) and every \( i \in \{0, 1, \ldots, n\} \), one has \( f_{m}^{i}(F_{m}^{i}) \subseteq F_{m}^{i-1} \). Moreover, the sequence

\[
0 \rightarrow F_{m-1}^{n} \oplus F_{m}^{n} \rightarrow \cdots \rightarrow F_{m-1}^{1} \oplus F_{m}^{1} \overset{f_{m}^{1}|_{F_{m}^{1}}}{\rightarrow} F_{m-1}^{0} \oplus F_{m}^{0} \overset{f_{m}^{0}|_{F_{m}^{0}}}{\rightarrow} X_{m}
\]

is exact.

Proof. Let \((0, b) \in F_{m}^{i}\). We have \( f_{m}^{i}(0, b) = f_{m}^{i} \circ \partial_{m+1}^{i}(b, 0) = \partial_{m+1}^{i-1} \circ f_{m+1}^{i}(b, 0) \in F_{m}^{i-1}\). It follows that \( \text{Im}(f_{m}^{i}|_{F_{m}^{i}}) \subseteq \text{Ker}(f_{m}^{i-1}|_{F_{m}^{i-1}}) \), for every \( m \in \mathbb{Z} \). To prove the other inclusion, we start with \( i = n \). Let \((0, b) \in F_{m}^{n}\) such that \((0, b) = f_{m}^{n}(0, b)\). Then there exists \((\alpha, \beta) \in F_{m-1}^{n} \oplus F_{m}^{n}\) such that \((0, b) = f_{m}^{n}(\alpha, \beta)\), since the \( m \)-th column is exact. On the other hand,

\[
f_{m-1}^{n}(0, \alpha) = f_{m-1}^{n} \circ \partial_{m}^{n}(\alpha, \beta) = \partial_{m}^{n-1} \circ f_{m}^{n}(\alpha, \beta) = \partial_{m}^{n-1}(0, b) = (0, 0).
\]

Since \( f_{m-1}^{n} \) is injective, we have \((0, \alpha) = (0, 0)\) and so \((0, b) = f_{m}^{n}(0, \beta)\), i.e. \( \text{Im}(f_{m}^{n}|_{F_{m}^{n}}) \) contains \( \text{Ker}(f_{m}^{n-1}|_{F_{m}^{n-1}}) \), for every \( m \in \mathbb{Z} \). Now we show that \( \text{Im}(f_{m}^{n-1}|_{F_{m}^{n-1}}) \) contains \( \text{Ker}(f_{m}^{n-2}|_{F_{m}^{n-2}}) \). Let \((0, b) \in \text{Ker}(f_{m}^{n-2}). Since the
central column is exact, there exists \((\alpha, \beta) \in F_{m-1}^n \oplus F_{m-1}^n\) such that \((0, b) = f_{m-1}^{-1}(\alpha, \beta)\). On the other hand,

\[
f_{m-1}^{-1}(0, \alpha) = f_{m-1}^{-1} \circ \partial_{m-1}^{-1}(\alpha, \beta) = \partial_{m-2}^{-1} \circ f_{m}^{-1}(\alpha, \beta) = \partial_{m-2}^{-1}(0, b) = (0, 0).
\]

Then \((0, \alpha) \in \text{Ker}(f_{m-1}^{-1}|_{F_{m-1}^n}) = \text{Im}(f_{m-1}^{-1}|_{F_{m-1}^n})\) and so there exists an element \((0, \gamma)\) in \(F_{m-2}^n \oplus F_{m-1}^n\) such that \((0, \alpha) = f_{m-1}^{-1}(0, \gamma)\). Since \((\alpha, \beta) - f_{m}^{-1}(\gamma, 0) \in F_{m-1}^{-1} \oplus F_{m-1}^{-1}\), we have

\[
\partial_{m-1}^{-1}((\alpha, \beta) - f_{m}^{-1}(\gamma, 0)) = (0, \alpha) - \partial_{m-1}^{-1} \circ f_{m}^{-1}(\gamma, 0) \\
= (0, \alpha) - f_{m-1}^{-1} \circ \partial_{m-2}^{-1}(\gamma, 0) \\
= (0, \alpha) - f_{m-1}^{-1}(0, \gamma) = (0, 0),
\]

i.e. \((\alpha, \beta) - f_{m}^{-1}(\gamma, 0) \in \text{Ker}(\partial_{m}^{-1}) = F_{m}^{-1}\). Also,

\[
f_{m}^{-1}((\alpha, \beta) - f_{m}^{-1}(\gamma, 0)) = f_{m}^{-1}(\alpha, \beta) = (0, b).
\]

Hence \(\text{Im}(f_{m}^{-1}|_{F_{m-n}^n}) \supseteq \text{Ker}(f_{m-2}^{-1}|_{F_{m-2}^n})\). Repeating the same argument several times, we get the exact sequence

\[
0 \longrightarrow F_{m-1}^n \oplus F_m^n \longrightarrow \cdots \longrightarrow F_{m-1}^1 \oplus F_m^1 \xrightarrow{f_{m}^{-1}|_{F_m^1}} F_{m-1}^0 \oplus F_m^0 \xrightarrow{f_{m}^0|_{F_m^0}} X_m.
\]

Given a chain complex \(X = (X_m, \partial^X_m)_m \in \mathbb{Z}\), its cardinal number is defined as \(\text{Card}(X) = \sum_{m \in \mathbb{Z}} \text{Card}(X_m)\). We shall say that \(X\) is small if \(\text{Card}(X) \leq \kappa\). We shall denote \(x \in X\) whenever there exists \(m \in \mathbb{Z}\) such that \(x \in X_m\).

**Lemma 2.3.4.** If \(X\) is an \(n\)-projective complex and \(x \in X\), then there exists a small \(n\)-projective subcomplex \(X' \subseteq X\) with \(x \in X'\), such that \(X/X'\) is also \(n\)-projective.

**Proof.** We only prove the case when \(n = 1\). The general case follows similarly. Consider a free resolution of \(X\) of length 1 in \(\text{Ch}(R)\):

\[
0 \longrightarrow L^1 \xrightarrow{f^1} L^0 \xrightarrow{f^0} X \longrightarrow 0.
\]

The idea of the proof is to apply a generalization of the zig-zag argument to produce small free subcomplexes of \(L^0\) and \(L^1\), say \(\mathcal{L}^0\) and \(\mathcal{L}^1\), and a short exact sequence of the form

\[
0 \longrightarrow \mathcal{L}^1 \xrightarrow{f^1|_{\mathcal{L}^1}} \mathcal{L}^0 \xrightarrow{f^0|_{\mathcal{L}^0}} X
\]

where \(X' = \text{CoKer}(f^1|_{\mathcal{L}^1})\) is a subcomplex of \(X\) with \(x \in X'\).
Since $L^0$ and $L^1$ are free complexes,

$$L^0 \cong \bigoplus_{i \in \mathbb{Z}} D^i(F^0_{i-1}) \quad \text{and} \quad L^1 \cong \bigoplus_{i \in \mathbb{Z}} D^i(F^1_{i-1}),$$

where $F^0_i$ and $F^1_i$ are free modules, for every $i \in \mathbb{Z}$. Let $B^0_i$ and $B^1_i$ be bases of $F^0_i$ and $F^1_i$, respectively. Then $B^0_{i-1} \cup B^0_i$ and $B^1_{i-1} \cup B^1_i$ are bases of $F^0_{i-1} \oplus F^0_i$ and $F^1_{i-1} \oplus F^1_i$, respectively. Suppose $x \in X_m$. At the $m$-th level, we have the following exact sequence

$$0 \rightarrow F^1_{m-1} \oplus F^1_m \xrightarrow{f^1_m} F^0_{m-1} \oplus F^0_m \xrightarrow{f^0_m} X_m \xrightarrow{\partial} 0.$$  

Consider the free resolution above as the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
0 & \rightarrow & F^1_m \oplus F^1_{m+1} & \rightarrow & F^1_{m-1} \oplus F^1_m & \rightarrow & F^1_{m-2} \oplus F^1_{m-1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & F^0_m \oplus F^0_{m+1} & \rightarrow & F^0_{m-1} \oplus F^0_m & \rightarrow & F^0_{m-2} \oplus F^0_{m-1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & X_{m+1} & \rightarrow & X_m & \rightarrow & X_{m-1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 
\end{array}
$$

Let $W^0_{m-1} \cup W^0_m$ be a finite subset of $B^0_{m-1} \cup B^0_m$ (we mean $W^0_{m-1} \subseteq B^0_{m-1}$ and $W^0_m \subseteq B^0_m$) such that $x \in f^0_m(<W^0_{m-1} \cup W^0_m>)$. Now let $W^1_{m-1} \cup W^1_m \subseteq B^1_{m-1} \cup B^1_m$ be a small set such that

$$f^1_m(<W^1_{m-1} \cup W^1_m>) \supseteq \text{Ker}(f^0_m|_{W^0_{m-1} \cup W^0_m}).$$

It is not necessarily true that $f^1_{m-1}(<W^1_{m-1}>) \supseteq \text{Ker}(f^0_{m-1}|_{W^0_{m-1}})$. Since the sequence

$$0 \rightarrow F^1_{m-1} \xrightarrow{f^1_{m-1}} F^0_{m-1} \xrightarrow{f^0_{m-1}} X_{m-1}$$

is exact, there exists a small set $\tilde{W}^1_{m-1} \subseteq B^1_{m-1}$ such that $f^1_{m-1}(<\tilde{W}^1_{m-1}>)$ does contain $\text{Ker}(f^0_{m-1}|_{W^0_{m-1}})$. Adding to $\tilde{W}^1_{m-1}$ the elements of $W^1_{m-1}$ which are not in $\tilde{W}^1_{m-1}$, we may assume that $\tilde{W}^1_{m-1} \supseteq W^1_{m-1}$. So we have
Moreover, the following diagram commutes and has exact rows and columns:

• \( f_m^{1}(<\tilde{W}_{m-1}^1 \cup W_m^1>) \supseteq f_m^{1}(<W_{m-1}^1 \cup W_m^1>) \supseteq \text{Ker}(f_m^{0}|_{<W_{m-1}^0 \cup W_m^0>}.

• \( f_{m-1}^{1}(<\tilde{W}_{m-1}^1>) \supseteq \text{Ker}(f_{m-1}^{0}|_{<W_{m-1}^0>}.

Summarizing, we can choose a small set \( W_{m-1}^1 \cup W_m^1 \subseteq B_{m-1}^1 \cup B_m^1 \) such that

• \( f_m^{1}(<W_{m-1}^1 \cup W_m^1>) \supseteq \text{Ker}(f_m^{0}|_{<W_{m-1}^0 \cup W_m^0>}.

• \( f_{m-1}^{1}(<W_{m-1}^1>) \supseteq \text{Ker}(f_{m-1}^{0}|_{<W_{m-1}^0>}.

Notice that \( <W_{m-1}^1 \cup W_m^1> = <W_{m-1}^1> \oplus <W_m^1> \), for \( j = 0, 1 \).

We go back to \( F_{m-1} \oplus F_m \). Choose a small set \( W_{m-1}^{0,(1)} \cup W_m^{0,(1)} \subseteq B_{m-1}^1 \cup B_m^1 \) containing \( W_{m-1}^0 \cup W_m^0 \) (and so \( W_{m-1}^{0,(1)} \supseteq W_{m-1}^0 \) and \( W_m^{0,(1)} \supseteq W_m^0 \)), such that

\[
\begin{align*}
\text{Ker}(f_m^0|_{<W_{m-1}^{0,(1)} \cup W_m^{0,(1)}>}) & 
\subseteq \left< W_{m-1}^{0,(1)} \right> \oplus \left< W_m^{0,(1)} \right>.
\end{align*}
\]

Note that \( f_{m-1}^{1}(<W_{m-1}^1>) \subseteq < W_{m-1}^{0,(1)}>.

Now we enlarge \( W_{m-1}^1 \cup W_m^1 \), i.e. we choose a small set \( W_{m-1}^{1,(1)} \cup W_m^{1,(1)} \subseteq B_{m-1}^1 \cup B_m^1 \) containing \( W_{m-1}^1 \cup W_m^1 \) such that

• \( f_m^{1}(<W_{m-1}^{1,(1)} \cup W_m^{1,(1)}) \supseteq \text{Ker}(f_m^{0}|_{<W_{m-1}^{0,(1)} \cup W_m^{0,(1)}>},

• \( f_{m-1}^{1}(<W_{m-1}^{1,(1)}> \supseteq \text{Ker}(f_{m-1}^{0}|_{<W_{m-1}^{0,(1)}>},

Then enlarge \( W_{m-1}^{0,(1)} \cup W_m^{0,(1)} \) to a small subset \( W_{m-1}^{0,(2)} \cup W_m^{0,(2)} \subseteq B_{m-1}^0 \cup B_m^0 \) and so on. For \( k = 0, 1, \) set

• \( B_{m-1}^{k,(0)} = \bigcup_{j=0}^\infty W_{m-1}^{k,(j)}, \) where \( W_{m-1}^{k,(0)} = W_{m-1}^k \) and \( W_{m-1}^{k,(1)} \subseteq W_{m-1}^{k,(1)} \subseteq \cdots ,

• \( B_m^{k,(0)} = \bigcup_{j=0}^\infty W_m^{k,(j)}, \) where \( W_m^{k,(0)} = W_m^k \) and \( W_m^{k,(1)} \subseteq W_m^{k,(1)} \subseteq \cdots .

Note that the sets above are linearly independent and small. By construction, we have

• \( f_m^{1}<B_{m-1}^{1,(0)} \oplus <B_{m}^{1,(0)}> \subseteq <B_{m-1}^{0,(0)} \oplus <B_{m}^{0,(0)}>, \) and

• \( f_{m-1}^{1}<B_{m-1}^{1,(0)} \subseteq <B_{m-1}^{0,(0)}>. \)

Moreover, the following diagram commutes and has exact rows and columns:
where the morphisms appearing in it are their corresponding restrictions. At this point, the problem is that we do not know if the $m + 1$-th row is exact. Actually, we do not even know if $f_{m+1}^1(<B_{1m}^{1,0}>) \subseteq <B_{0m}^{0,0}>$. In order to fix this problem, we are going to refine the sets of generators just obtained applying the zig-zag argument again, without destroying exactness in the other rows.

Choose a small set $Y_m^0 \sqcup Y_{m+1}^0 \subseteq B_m^0 \sqcup B_{m+1}^0$ containing $B_m^{0,0}$ (and so $B_m^{0,0} \subseteq Y_m^0$) such that

$$f_{m+1}^1 \left( \langle B_{m-1}^{1,0} \rangle \right) \subseteq \langle Y_m^0 \rangle \oplus \langle Y_{m+1}^0 \rangle.$$ 

Note that

$$f_m^1 \left( \langle B_{m-1}^{1,0} \rangle \oplus \langle B_m^{1,0} \rangle \right) \subseteq \langle B_{m-1}^{0,0} \rangle \oplus \langle Y_m^0 \rangle.$$ 

Now choose a small set $Y_m^1 \sqcup Y_{m+1}^1 \subseteq B_m^1 \sqcup B_{m+1}^1$ containing $B_m^{1,0}$ (i.e. $B_m^{1,0} \subseteq Y_m^1$) such that

$$f_{m+1}^1 \left( \langle Y_m^1 \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m+1}^0 \langle Y_m^0 \rangle \oplus \langle Y_{m+1}^0 \rangle \right).$$

It is not necessarily true that

$$f_m^1 \left( \langle B_{m-1}^{1,0} \rangle \oplus \langle Y_m^1 \rangle \right) \supseteq \text{Ker} \left( f_m^0 \langle B_{m-1}^{0,0} \rangle \oplus \langle Y_m^0 \rangle \right).$$

But there exists a small set $\tilde{Y}_{m-1}^1 \sqcup \tilde{Y}_m^1 \subseteq B_{m-1}^1 \sqcup B_m^1$ containing $B_{m-1}^{1,0} \sqcup B_m^{1,0}$.
such that
\[ f_m^1 \left( \langle Y_{m-1}^1 \rangle \oplus \langle Y_m^1 \rangle \right) \supseteq \text{Ker} \left( f_{m+1}^0 \big|_{B_m^{0,(0)}} \right). \]

We may assume that \( Y_1^1 \supseteq Y_m^1 \). So
\[ f_{m+1}^1 \left( \langle Y_{m-1}^1 \rangle \oplus \langle Y_m^1 \rangle \right) \supseteq \text{Ker} \left( f_{m+1}^0 \big|_{(Y_m^1)\oplus(Y_m^1)} \right). \]

Note that
\[ f_{m-1}^1 \left( \langle Y_{m-1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m-1}^0 \big|_{B_{m-1}^{0,(0)}} \right). \]

Summarizing, we may choose small sets \( Y_{m-1}^1 \subseteq B_{m-1}^1 \), \( Y_m^1 \subseteq B_m^1 \), \( Y_{m+1}^1 \subseteq B_{m+1}^1 \), containing \( B_{m-1}^{1,(0)} \), \( B_m^{1,(0)} \) and \( B_{m+1}^{1,(0)} \), respectively, such that
- \( f_{m+1}^1(Y_m^1) > \oplus < Y_{m+1}^1 \supseteq \text{Ker}(f_{m+1}^0|_{B_m^{0,(0)}}) \),
- \( f_m^1(Y_{m-1}^1) > \oplus < Y_m^1 \supseteq \text{Ker}(f_m^0|_{B_{m-1}^{0,(0)}}) \),
- \( f_{m-1}^1(Y_{m-1}^1) > \supseteq \text{Ker}(f_{m-1}^0|_{B_{m-1}^{0,(0)}}) \).

Now choose a small set \( Y_m^{0,(1)} \cup Y_{m+1}^{0,(1)} \subseteq B_m^0 \cup B_{m+1}^0 \) containing \( Y_m^0 \cup Y_{m+1}^0 \) such that
\[ f_{m+1}^1(Y_m^1 \oplus Y_{m+1}^1) \subseteq \langle Y_{m-1}^0 \rangle \oplus \langle Y_{m+1}^1 \rangle. \]

On the other hand, choose a small set \( B_{m-1}^{0,(0)} \cup B_m^{0,(1)} \subseteq B_{m-1}^0 \cup B_m^0 \) such that
\[ f_{m}^1(Y_{m-1}^1 \oplus Y_m^1) \subseteq \langle Y_{m-1}^0 \rangle \oplus \langle Y_m^1 \rangle. \]

Note that \( f_{m-1}^1(Y_{m-1}^1) \subseteq < Y_{m-1}^0 \rangle \) and that we may choose \( Y_m^{0,(1)} \) containing \( Y_m^{0,(1)} \). Summarizing, there exist small sets \( Y_m^{0,(1)} \subseteq B_m^1 \), \( Y_{m+1}^{0,(1)} \subseteq B_m^0 \) and \( Y_{m+1}^{0,(1)} \subseteq B_{m+1}^0 \) containing \( B_m^{0,(0)} \), \( Y_m^0 \) and \( Y_{m+1}^0 \), respectively, such that
- \( f_{m+1}^1(Y_m^1) > \oplus < Y_{m+1}^1 \supseteq < Y_{m+1}^0 \rangle \),
- \( f_m^1(Y_{m-1}^1) > \oplus < Y_m^1 \supseteq < Y_m^0 \rangle \),
- \( f_{m-1}^1(Y_{m-1}^1) > c < Y_{m-1}^0 \rangle \).

Now choose small sets \( Y_{m-1}^{1,(1)} \subseteq B_m^{1,(1)} \), \( Y_m^{1,(1)} \subseteq B_m^{1,(1)} \) and \( Y_{m+1}^{1,(1)} \subseteq B_{m+1}^{1,(1)} \) containing \( Y_{m-1}^1 \), \( Y_m^1 \) and \( Y_{m+1}^1 \), respectively, such that
- \( f_{m+1}^1(Y_{m+1}^{1,(1)}) > \oplus < Y_{m+1}^{1,(1)} \rangle \supseteq \text{Ker}(f_{m+1}^0|_{B_{m+1}^{0,(1)}}) \),
- \( f_m^1(Y_{m-1}^{1,(1)}) > \oplus < Y_{m-1}^{1,(1)} \rangle \supseteq \text{Ker}(f_m^0|_{B_{m-1}^{0,(1)}}) \).

13
It is not necessarily true that
\[ f_{m-1}^{1}(Y_{m-1}^{1,1}) \supseteq \ker \left( f_{m-1}^{0} \big|_{Y_{m-1}^{0,1}} \right). \]

But using similar arguments as above, we can enlarge \( Y_{m-1}^{1,1} \) in such a way that the previous inclusion is satisfied.

Continue using the zig-zag procedure to enlarge \( Y_{m}^{0,1} \), \( Y_{m}^{0,1} \), \( Y_{m+1}^{0,1} \) and so on. Then set
- \( B_{m-1}^{0,1} = Y_{m-1}^{0,1} \cup Y_{m-1}^{0,2} \cup \cdots \),
- \( B_{m-1}^{1,1} = Y_{m-1}^{1,1} \cup Y_{m-1}^{1,2} \cup \cdots \),
- \( B_{m}^{0,1} = Y_{m}^{0,1} \cup Y_{m}^{0,2} \cup \cdots \),
- \( B_{m}^{1,1} = Y_{m}^{1,1} \cup Y_{m}^{1,2} \cup \cdots \),
- \( B_{m+1}^{0,1} = Y_{m+1}^{0,1} \cup Y_{m+1}^{0,2} \cup \cdots \),
- \( B_{m+1}^{1,1} = Y_{m+1}^{1,1} \cup Y_{m+1}^{1,2} \cup \cdots \).

Hence we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
\langle B_{m+1}^{1,1} \rangle \oplus \langle 0 \rangle & \langle 0 \rangle \\
\downarrow \partial_{m+2}^{1} & \\
0 & \langle B_{m}^{0,1} \rangle \oplus \langle B_{m+1}^{0,1} \rangle \\
\downarrow \partial_{m+1}^{1} & \downarrow f_{m+1}^{1} \\
\langle B_{m}^{1,1} \rangle \oplus \langle B_{m+1}^{1,1} \rangle & \langle B_{m}^{0,1} \rangle \oplus \langle B_{m+1}^{0,1} \rangle \\
\downarrow \partial_{m}^{1} & \downarrow \partial_{m+1}^{0} \\
0 & \langle B_{m+1}^{0,1} \rangle \oplus \langle B_{m+1}^{0,1} \rangle \\
\downarrow f_{m}^{1} & \downarrow \partial_{m}^{0} \\
\langle B_{m-1}^{1,1} \rangle \oplus \langle B_{m-1}^{1,1} \rangle & \langle B_{m-1}^{0,1} \rangle \oplus \langle B_{m-1}^{0,1} \rangle \\
\downarrow \partial_{m-1}^{1} & \downarrow \partial_{m-1}^{0} \\
0 & 0 \\
\vdots & \vdots \\
\end{array}
\]
As in the previous iteration, we do not know if
\[ f_{m+2}^1 \left( \langle B_{m+i}^{1,(1)} \rangle \oplus \langle 0 \rangle \right) \subseteq \langle B_{m+i}^{0,(1)} \rangle \oplus \langle 0 \rangle. \]

Then repeat the same process and so on. In the \( i \)-th iteration, we get, for \( j = 0, 1 \), small sets \( B_j^{i,(k)} \) such that:

- \( B_j^{i,(i)} \subseteq B_j^i \),
- \( B_j^{i,(i-1)} \subseteq B_j^{i-1} \subseteq B_j^i \),
- \( B_j^{i,(i-2)} \subseteq B_j^{i-2} \subseteq B_j^{i-1} \subseteq B_j^i \),
- \( \ldots \)
- \( B_j^{i,(0)} \subseteq \cdots \subseteq B_j^{i,-1} \subseteq B_j^0 \subseteq B_j^i \).

and the following commutative diagram with exact rows and columns:

Finally, for \( j = 0, 1 \), set:
\[\begin{align*}
B_{m-1}^j &= B_{m-1}^{j,(0)} \cup B_{m-1}^{j,(1)} \cup \cdots, \\
B_m^j &= B_m^{j,(0)} \cup B_m^{j,(1)} \cup \cdots, \\
B_{m+1}^j &= B_{m+1}^{j,(1)} \cup B_{m+1}^{j,(2)} \cup \cdots, \\
B_{m+i}^j &= B_{m+i}^{j,(i)} \cup B_{m+i}^{j,(i+1)} \cup \cdots,
\end{align*}\]

All of these sets are small. We have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & \to & \langle B_{m+i}^1 \rangle \oplus \langle B_{m+i+1}^1 \rangle \\
\downarrow & & \downarrow \\
0 & \to & \langle B_{m-1}^1 \rangle \oplus \langle B_m^1 \rangle \\
\downarrow & & \downarrow \\
0 & \to & \langle 0 \rangle \oplus \langle B_{m-1}^1 \rangle \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\downarrow & & \downarrow \\
& & \\
\end{array}
\]

0 \to \langle B_{m+i}^1 \rangle \oplus \langle B_{m+i+1}^1 \rangle \xrightarrow{f_{m+i+1}} \langle B_{m+i}^0 \rangle \oplus \langle B_{m+i+1}^0 \rangle \xrightarrow{f_{m+i}} X_{m+i+1}

\[
\begin{array}{ccc}
\downarrow & & \\
\downarrow & & \\
\downarrow & & \\
\downarrow & & \\
\downarrow & & \\
\end{array}
\]

\[
\begin{array}{ccc}
\partial_{m+i+1} & & \\
\partial_0 & & \\
\partial_{m+i+1} & & \\
\partial_0 & & \\
\partial_{m+i+1} & & \\
\partial_0 & & \\
\end{array}
\]

\[
0 \to \langle B_{m-1}^1 \rangle \oplus \langle B_m^1 \rangle \xrightarrow{f_m} \langle B_{m-1}^0 \rangle \oplus \langle B_m^0 \rangle \xrightarrow{f_m} X_m
\]

\[
0 \to \langle 0 \rangle \oplus \langle B_{m-1}^1 \rangle \xrightarrow{f_{m-1}} \langle 0 \rangle \oplus \langle B_{m-1}^0 \rangle \xrightarrow{f_{m-1}} X_{m-1}
\]

\[
0 \to 0 \xrightarrow{f_{m}} X_{m-2}
\]

We have obtained the following exact sequence in \(\text{Ch}(R)\):

\[
0 \to \bigoplus_{i \geq m} D^i \langle (B_{i-1}^1) \rangle \xrightarrow{f_{i-1}(\oplus_{i \geq m} \sigma^i(\langle B_{i-1}^1 \rangle))} \bigoplus_{i \geq m} D^i \langle (B_{i-1}^0) \rangle
\]

\[
X' \xrightarrow{f_i(\oplus_{i \geq m} \sigma^i(\langle B_{i-1}^0 \rangle))} 0
\]
where $X' = \text{CoKer}(f^1|_{\bigoplus_{i \geq m} D^i(<B_{i-1}^1>)}).$ Note that $x \in X'$, that $X'$ is a subcomplex of $X$, and that $\bigoplus_{i \geq m} D^i(<B_{i-1}^1>)$ and $\bigoplus_{i \geq m} D^i(<B_{i-1}^0>)$ are small subcomplexes of $L^0$ and $L^1$, respectively. Since $f^0|_{\bigoplus_{i \geq m} D^i(<B_{i-1}^0>)}$ is surjective, we have that $X'$ is also small. The exact sequence above is a projective resolution of $X'$ of length 1, hence $X'$ is 1-projective.

Now consider the following commutative diagram

$\begin{array}{cccccc}
0 & \bigoplus_{i \geq m} D^i(<B_{i-1}^1>) & \bigoplus_{i \geq m} D^i(<B_{i-1}^0>) & \longrightarrow & X' & \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & L^1 & f^1 & \longrightarrow & L^0 & \longrightarrow X \longrightarrow 0
\end{array}$

Taking the quotient of the resolution of $X$ by the resolution of $X'$, we get a free resolution of $X/X'$, and so $X/X'$ is also 1-projective. □

Given a chain complex $X \in \text{Ch}(R)$ and an ordinal number $\lambda$, a filtration of $X$ indexed by $\lambda$ is a family $(X_\alpha)_{\alpha \leq \lambda}$ of subcomplexes of $X$ such that

(a) $X_\lambda = X$,
(b) $X_0 = 0$,
(c) $X_\alpha$ is a subcomplex of $X_{\alpha'}$ whenever $\alpha \leq \alpha'$, and
(d) $X_\beta = \bigcup_{\alpha < \beta} X_\alpha$ whenever $\beta$ is a limit ordinal.

By the union of chain complexes $X_\beta = \bigcup_{\alpha < \beta} X_\alpha$ we mean the chain complex whose objects are given by

$$X_{\beta,n} = \bigcup_{\alpha < \beta} X_{\alpha,n} = \{(\alpha, x) : \alpha < \beta \text{ and } x \in X_{\alpha,n}\},$$

and whose morphisms are given by

$$\partial_n^{X_\beta} : X_{\beta,n} \longrightarrow X_{\beta,n-1}$$

$$(\alpha, x) \mapsto (\alpha, \partial_n^{X_\alpha}(x)).$$

If $S$ is some class of complexes in $\text{Ch}(R)$, we shall say that a filtration $(X_\alpha)_{\alpha \leq \lambda}$ of $X$ is an $S$-filtration if for each $\alpha + 1 < \lambda$ we have that $X_0$ and $X_{\alpha+1}/X_\alpha$ are isomorphic to elements of $S$.

**Lemma 2.3.5** (Eklof). Let $X$ and $Y$ be chain complexes and let $(X_\alpha)_{\alpha \leq \lambda}$ be a $\bot\{Y\}$-filtration of $X$. Then $\text{Ext}^1(X,Y) = 0$. 

17
The proof of the previous result is given in [3, Theorem 7.3.4] in the category \( R\text{-Mod} \) carries over directly to the category \( \text{Ch}(R) \).

**Theorem 2.3.1.** The cotorsion pair \((\widetilde{P}_n, \widetilde{P}_n^\perp)\) is complete.

**Proof.** Applying Lemma 2.3.4, we have that any complex \( X \in \widetilde{P}_n \) can be written as a union \( X = \bigcup_{\alpha<\lambda} X_\alpha \), where \((X_\alpha)_{\alpha<\lambda}\) is a \( \widetilde{P}_n \)-filtration, and \( X_0 \) and \( X_{\alpha+1}/X_\alpha \) are small complexes whenever \( \alpha+1<\lambda \). We construct such a filtration by using transfinite induction. Set \( X_0 = 0 \). Now choose any \( x \neq 0 \) in \( X \) and let \( X_1 \) be the complex given by Lemma 2.3.4. We have that \( X_0, X_1 \in \widetilde{P}_n \) and they are small. Also, \( X/X_1 \in \widetilde{P}_n \). If \( X_1 \subseteq X \) then choose \( x' + X_1 \neq 0 + X_1 \) in \( X/X_1 \). Using Lemma 2.3.4 again, we can construct a chain complex \( X_2/X_1 \subseteq X/X_1 \) such that \( x' + X_1 \in X_2/X_1 \), \( X_2/X_1 \) is small and \( X_2/X_1 \in \widetilde{P}_n \). Now suppose that \( \beta \) is an ordinal and that for any \( \alpha<\beta \) one has constructed chain complexes \( X_\alpha \subseteq X \) such that:

(a) \( X_\alpha \subseteq X_\alpha' \) whenever \( \alpha \leq \alpha' \),

(b) \( X_{\alpha+1}/X_\alpha \in \widetilde{P}_n \) and \( X_{\alpha+1}/X_\alpha \) is small whenever \( \alpha+1<\beta \),

(c) \( X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha \) for every limit ordinal \( \gamma < \beta \).

If \( \beta \) is a limit ordinal, set \( X_\beta = \bigcup_{\alpha<\beta} X_\alpha \). Otherwise there exists an ordinal \( \alpha<\beta \) such that \( \beta = \alpha+1 \). Then construct \( X_{\alpha+1} \) from \( X_\alpha \) as we just constructed \( X_2 \) from \( X_1 \). By transfinite induction, we have a \( \widetilde{P}_n \)-filtration \((X_\alpha : \alpha < \lambda)\) of \( X \) such that \( X_{\alpha+1}/X_\alpha \) is small whenever \( \alpha+1<\lambda \). Now consider the set

\[ S = \left\{ L \in \widetilde{P}_n \mid L \text{ is small complex} \right\}. \]

Note that \( \widetilde{P}_n^\perp \subseteq S^\perp \) since \( S \subseteq \widetilde{P}_n \). Now let \( Y \in S^\perp \) and \( X \in \widetilde{P}_n \). Write \( X = \bigcup_{\alpha<\lambda} X_\alpha \) with \((X_\alpha)_{\alpha<\lambda}\) as above. Then \( X_0, X_{\alpha+1}/X_\alpha \in S \). Hence \( \text{Ext}^1(X_0, Y) = 0 \) and \( \text{Ext}^1(X_{\alpha+1}/X_\alpha, Y) = 0 \), i.e. \((X_\alpha)_{\alpha<\lambda}\) is a \( ^\perp \{Y\} \)-filtration of \( X \). By the Eklof Lemma, we have \( \text{Ext}^1(X, Y) = 0 \) and so \( S^\perp \subseteq \widetilde{P}_n^\perp \). We have \( \widetilde{P}_n^\perp = S^\perp \). Therefore, by the Eklof and Trlifaj Theorem we conclude that \((\widetilde{P}_n, \widetilde{P}_n^\perp)\) is a complete cotorsion pair. \( \square \)

**References**


